

Derivation of Gravitational Self-Force

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Abstract

We analyze the issue of “particle motion” in general relativity in a systematic and rigorous way by considering a one-parameter family of metrics corresponding to having a body (or black hole) that is “scaled down” to zero size and mass in an appropriate manner. We prove that the limiting worldline of such a one-parameter family must be a geodesic of the background metric and obtain the leading order perturbative corrections, which include gravitational self-force, spin force, and geodesic deviation effects. The status the MiSaTaQuWa equation is explained as a candidate “self-consistent perturbative equation” associated with our rigorous perturbative result.

It is of considerable interest to determine the motion of a body in general relativity in the limit of small size, taking into account the deviations from geodesic motion arising from gravitational self-force effects. There is a general consensus that the gravitational self-force is given by the “MiSaTaQuWa equations”: In the absence of incoming radiation, the motion is given by

$$u^\nu \nabla_\nu u^\mu = -\frac{1}{2}(g^{\mu\nu} + u^\mu u^\nu)(2\nabla_\sigma h_{\nu\rho}^{\text{tail}} - \nabla_\nu h_{\rho\sigma}^{\text{tail}})|_{z(\tau)} u^\rho u^\sigma, \quad (1)$$

$$h_{\mu\nu}^{\text{tail}}(x) = M \int_{-\infty}^{\tau^-} \left(G_{\mu\nu\mu'\nu'}^+ - \frac{1}{2} g_{\mu\nu} G_{\rho\mu'\nu'}^+ \right) (x, z(\tau')) u^{\mu'} u^{\nu'} d\tau', \quad (2)$$

where G_+ is the retarded Green’s function for the wave operator $\nabla^\alpha \nabla_\alpha \tilde{h}_{\mu\nu} - 2R^\alpha{}_{\mu\nu}{}^\beta \tilde{h}_{\alpha\beta}$. (Note that the τ^- limit of integration indicates that only the part of G_+ interior to the light cone contributes to $h_{\mu\nu}^{\text{tail}}$.) However, all derivations contain some unsatisfactory features. This is not surprising in view of the fact that, as noted in [1], “point particles” do not make sense in nonlinear theories like general relativity!

- Derivations that treat the body as a point particle require unjustified “regularizations”.
- Derivations using matched asymptotic expansions [2] make a number of ad hoc and/or unjustified assumptions.
- The axioms of the Quinn-Wald axiomatic approach [3] have not been shown to follow from Einstein’s equation.
- All of the above derivations employ at some stage a “phoney” version of the linearized Einstein equation with a point particle source, wherein the Lorenz gauge version of the linearized Einstein equation is written down, but the Lorenz gauge condition is not imposed.

How should gravitational self-force be rigorously derived? A precise formula for gravitational self-force can hold only in a limit where the size, R , of the body goes to zero. Since “point-particles” do not make sense in general relativity—collapse to a black hole would occur before a point-particle limit could be taken—the mass, M , of the body must also go to zero as $R \rightarrow 0$. In the limit as $R, M \rightarrow 0$, the worldtube of the body should approach a curve, γ , which should be a geodesic of the “background metric”. The self-force should arise as the lowest order in M correction to γ . In the following, we shall describe an approach

that we have recently taken to derive gravitational self-force in this manner. Details can be found in [4].

The discussion above suggests that we consider a one-parameter family of solutions to Einstein's equation, $(g_{\mu\nu}(\lambda), T_{\mu\nu}(\lambda))$, with $R(\lambda) \rightarrow 0$ and $M(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$. But, what conditions should be imposed on $(g_{\mu\nu}(\lambda), T_{\mu\nu}(\lambda))$ to ensure that it corresponds to a body that is shrinking down to zero size, but is not undergoing wild oscillations, drastically changing its shape, or doing other crazy things as it does so?

As a very simple, explicit example of the kind of one-parameter family we seek, consider the Schwarzschild-deSitter metrics with $M = \lambda$,

$$ds^2(\lambda) = -(1 - \frac{2\lambda}{r} - Cr^2)dt^2 + (1 - \frac{2\lambda}{r} - Cr^2)^{-1}dr^2 + r^2d\Omega^2. \quad (3)$$

If we take the limit as $\lambda \rightarrow 0$ at fixed coordinates (t, r, θ, ϕ) with $r > 0$, it is easily seen that we obtain the deSitter metric—with the deSitter spacetime worldline γ defined by $r = 0$ corresponding to the location of the black hole “before it disappeared”. However, there is also another limit that can be taken. At each time t_0 , one can “blow up” the metric $g_{\mu\nu}(\lambda)$ by multiplying it by λ^{-2} , i.e., define

$$\bar{g}_{\mu\nu}(\lambda) \equiv \lambda^{-2}g_{\mu\nu}(\lambda). \quad (4)$$

We correspondingly rescale the coordinates by defining $\bar{r} = r/\lambda$, $\bar{t} = (t - t_0)/\lambda$. Then

$$d\bar{s}^2(\lambda) = -(1 - 2/\bar{r} - \lambda^2 C \bar{r}^2)d\bar{t}^2 + (1 - 2/\bar{r} - \lambda^2 C \bar{r}^2)^{-1}d\bar{r}^2 + \bar{r}^2d\Omega^2 \quad (5)$$

In the limit as $\lambda \rightarrow 0$ (at fixed $(\bar{t}, \bar{r}, \theta, \phi)$) the “deSitter background” becomes irrelevant. The limiting metric is simply the Schwarzschild metric of unit mass. The fact that the limit as $\lambda \rightarrow 0$ exists can be attributed to the fact that the Schwarzschild black hole is shrinking to zero in a manner where, in essence, nothing changes except the overall scale.

The simultaneous existence of both of the above types of limits characterizes the type of one-parameter family of spacetimes $g_{\mu\nu}(\lambda)$ that we wish to consider. More precisely, we wish to consider a one parameter family of solutions $g_{\mu\nu}(\lambda)$ satisfying the following properties:

- (i) Existence of the “ordinary limit”: There exist coordinates x^α such that $g_{\mu\nu}(\lambda, x^\alpha)$ is jointly smooth in (λ, x^α) , at least for $r > \bar{R}\lambda$ for some constant \bar{R} , where $r \equiv \sqrt{\sum (x^i)^2}$ ($i = 1, 2, 3$). For all λ and for $r > \bar{R}\lambda$, $g_{\mu\nu}(\lambda)$ is a vacuum solution of Einstein's equation. Furthermore, $g_{\mu\nu}(\lambda = 0, x^\alpha)$ is smooth in x^α , including at $r = 0$, and, for $\lambda = 0$, the curve γ defined by $r = 0$ is timelike.

- (ii) Existence of the “scaled limit”: For each t_0 , we define $\bar{t} \equiv (t - t_0)/\lambda$, $\bar{x}^i \equiv x^i/\lambda$. Then the metric $\bar{g}_{\bar{\mu}\bar{\nu}}(\lambda; t_0; \bar{x}^\alpha) \equiv \lambda^{-2}g_{\mu\nu}(\lambda; t_0; \bar{x}^\alpha)$ is jointly smooth in $(\lambda, t_0; \bar{x}^\alpha)$ for $\bar{r} \equiv r/\lambda > \bar{R}$.

The above two conditions must be supplemented by an additional “uniformity requirement”, which can be explained as follows. From the definitions of $\bar{g}_{\bar{\mu}\bar{\nu}}$ and \bar{x}^μ , we can relate coordinate components of the barred metric in barred coordinates to coordinate components of the unbarred metric in corresponding unbarred coordinates,

$$\bar{g}_{\bar{\mu}\bar{\nu}}(\lambda; t_0; \bar{t}, \bar{x}^i) = g_{\mu\nu}(\lambda; t_0 + \lambda\bar{t}, \lambda\bar{x}^i). \quad (6)$$

Now introduce new variables $\alpha \equiv r$ and $\beta \equiv \lambda/r = 1/\bar{r}$, and view the metric components $g_{\mu\nu}(\lambda)$ as functions of $(\alpha, \beta, t, \theta, \phi)$, where θ and ϕ are defined in terms of x^i by the usual formula for spherical polar angles. We have

$$\bar{g}_{\bar{\mu}\bar{\nu}}(\alpha\beta, t_0; \bar{t}, 1/\beta, \theta, \phi) = g_{\mu\nu}(\alpha\beta, t = t_0 + \lambda\bar{t}; \alpha, \theta, \phi). \quad (7)$$

Then, by assumption (ii) we see that for $0 < \beta < 1/\bar{R}$, $g_{\mu\nu}$ is smooth in (α, β) for all α *including* $\alpha = 0$. By assumption (i), we see that for all $\alpha > 0$, $g_{\mu\nu}$ is smooth in (α, β) for $\beta < 1/\bar{R}$, *including* $\beta = 0$. Furthermore, for $\beta = 0$, $g_{\mu\nu}$ is smooth in α , *including* $\alpha = 0$.

We now impose the additional uniformity requirement on our one-parameter family of spacetimes:

- (iii) $g_{\mu\nu}$ is jointly smooth in (α, β) at $(0, 0)$.

We already know from our previous assumptions that $g_{\mu\nu}(\lambda; t_0, r, \theta, \phi)$ and its derivatives with respect to x^α approach a limit if we let $\lambda \rightarrow 0$ at fixed r and then let $r \rightarrow 0$. The uniformity requirement implies that the same limits are attained whenever λ and r both go to zero in any way such that λ/r goes to zero.

It has recently been proven in [5] that an analog of the uniformity requirement holds for electromagnetism in Minkowski spacetime in the following sense: Consider a one-parameter family of charge-current sources of the form $J^\mu(\lambda, t, x^i) = \tilde{J}^\mu(\lambda, t, x^i/\lambda)$ where \tilde{J}^μ is a smooth function of its arguments and $x^i = 0$ defines a timelike worldline. Then the retarded solution, $F_{\mu\nu}(\lambda, x^\mu)$, is a smooth function the variables $(\alpha, \beta, t, \theta, \phi)$ in a neighborhood of $(\alpha, \beta) = (0, 0)$. In the gravitational case, we do not have a simple relationship between the metric and the

stress-energy source, and in the nonlinear regime, it would not make sense to formulate the uniformity condition in terms of the behavior of the stress-energy. Consequently, we have formulated this condition in terms of the behavior of the metric itself.

The uniformity requirement implies that the metric components can be approximated near $(\alpha, \beta) = (0, 0)$ with a finite Taylor series in α and β ,

$$g_{\mu\nu}(\lambda; t, r, \theta, \phi) = \sum_{n=0}^N \sum_{m=0}^M r^n \left(\frac{\lambda}{r}\right)^m (a_{\mu\nu})_{nm}(t, \theta, \phi), \quad (8)$$

where remainder terms have been dropped. This gives a *far zone expansion*. Equivalently, we have

$$\bar{g}_{\bar{\mu}\bar{\nu}}(\lambda; t_0; \bar{t}, \bar{r}, \theta, \phi) = \sum_{n=0}^N \sum_{m=0}^M (\lambda \bar{r})^n \left(\frac{1}{\bar{r}}\right)^m (a_{\mu\nu})_{nm}(t_0 + \lambda \bar{t}, \theta, \phi). \quad (9)$$

Further Taylor expanding this formula with respect to the time variable yields a *near zone expansion*. Note that since we can express $\bar{g}_{\bar{\mu}\bar{\nu}}$ at $\lambda = 0$ as a series in $1/\bar{r}$ as $\bar{r} \rightarrow \infty$ and since $\bar{g}_{\bar{\mu}\bar{\nu}}$ at $\lambda = 0$ does not depend on \bar{t} , we see that $\bar{g}_{\bar{\mu}\bar{\nu}}(\lambda = 0)$ is a stationary, asymptotically flat spacetime.

The curve γ to which our body shrinks as $\lambda \rightarrow 0$ (see condition (i) above) can now be proven to be a geodesic of the metric $g_{\mu\nu}(\lambda = 0)$ as follows: Choose the coordinates x^α so that at $\lambda = 0$ they correspond to Fermi normal coordinates about the worldline γ . In particular, we have $g_{\mu\nu} = \eta_{\mu\nu}$ on γ at $\lambda = 0$. It follows from (8) that near γ (i.e., for small r) the metric $g_{\mu\nu}$ must take the form

$$g_{\mu\nu} = \eta_{\mu\nu} + O(r) + \lambda \left(\frac{C_{\mu\nu}(t, \theta, \phi)}{r} + O(1) \right) + O(\lambda^2) \quad (10)$$

Now, for $r > 0$, the coefficient of λ , namely

$$h_{\mu\nu} = \frac{C_{\mu\nu}}{r} + O(1) \quad (11)$$

must satisfy the vacuum linearized Einstein equation off of the background spacetime $g_{\mu\nu}(\lambda = 0)$. However, since each component of $h_{\mu\nu}$ is a locally L^1 function, it follows immediately that $h_{\mu\nu}$ is well defined as a distribution. It is not difficult to show that, as a distribution, $h_{\mu\nu}$ satisfies the linearized Einstein equation with source of the form $N_{\mu\nu}(t)\delta^{(3)}(x^i)$, where $N_{\mu\nu}$ is given by a formula involving the limit as $r \rightarrow 0$ of the angular average of $C_{\mu\nu}$ and its first derivative. The linearized Bianchi identity then immediately implies that $N_{\mu\nu}$ is of the form $M u_\mu u_\nu$ with M constant, and that γ is a geodesic for $M \neq 0$.

Our main interest, however, is not to rederive geodesic motion but to find the leading order corrections to geodesic motion that arise from finite mass and finite size effects. To define these corrections, we need to have a notion of the “location” of the body to first order in λ . This can be defined as follows: Since $\bar{g}_{\bar{\mu}\bar{\nu}}(\lambda = 0)$ is an asymptotically flat spacetime, its mass dipole moment can be set to zero (at all t_0) as a gauge condition on the coordinates \bar{x}^i . The new coordinates \bar{x}^i then have the interpretation of being “center of mass coordinates” for the spacetime $\bar{g}_{\bar{\mu}\bar{\nu}}(\lambda = 0)$. In terms of our original coordinates x^α , the transformation to center of mass coordinates at all t_0 corresponds to a coordinate transformation $x^\alpha \rightarrow \hat{x}^\alpha$ of the form

$$\hat{x}^\alpha(t) = x^\alpha - \lambda A^\alpha(t, x^i) + O(\lambda^2). \quad (12)$$

To first order in λ , the world line defined by $\hat{x}^i = 0$ should correspond to the “position” of the body. The first-order displacement from γ in the original coordinates is then given simply by

$$Z^i(t) \equiv A^i(t, x^j = 0). \quad (13)$$

The quantity Z^i is most naturally interpreted as a “deviation vector field” defined on γ . Our goal is to derive relations (if any) that hold for Z^i that are independent of the choice of one-parameter family satisfying our assumptions.

We now choose the x^α coordinates—previously chosen to agree with Fermi normal coordinates on γ at $\lambda = 0$ —to correspond to the Lorenz/harmonic gauge to first order in λ . To order λ^2 , the leading order in r terms in $g_{\alpha\beta}$ are,

$$\begin{aligned} g_{\alpha\beta}(\lambda; t, x^i) = & \eta_{\alpha\beta} + B_{\alpha i \beta j}(t) x^i x^j + O(r^3) \\ & + \lambda \left(\frac{2M}{r} \delta_{\alpha\beta} + h_{\alpha\beta}^{\text{tail}}(t, 0) + h_{\alpha\beta i}^{\text{tail}}(t, 0) x^i + M \mathcal{R}_{\alpha\beta}(t) + O(r^2) \right) \\ & + \lambda^2 \left(\frac{M^2}{r^2} (-2t_\alpha t_\beta + 3n_\alpha n_\beta) + \frac{2}{r^2} P_i(t) n^i \delta_{\alpha\beta} + \frac{1}{r^2} t_{(\alpha} S_{\beta)j}(t) n^j \right. \\ & \left. + \frac{1}{r} K_{\alpha\beta}(t, \theta, \phi) + H_{\alpha\beta}(t, \theta, \phi) + O(r) \right) + O(\lambda^3) \end{aligned} \quad (14)$$

Here B and \mathcal{R} are expressions involving the curvature of $g_{\mu\nu}(\lambda = 0)$ and we have introduced the “unknown” tensors K and H . The quantities P^i and $S_{\alpha\beta}$ turn out to be the mass dipole and spin of the “near-zone” background spacetime $\bar{g}_{\bar{\mu}\bar{\nu}}(\lambda = 0)$. For simplicity, we have assumed no “incoming radiation”. Hadamard expansion techniques and 2nd order perturbation theory were used to derive this expression.

Using the coordinate shift $x^\mu \rightarrow x^\mu - \lambda A^\mu$ to cancel the mass dipole term, the above expression translates into the following expression for the scaled metric

$$\begin{aligned} \bar{g}_{\bar{\alpha}\bar{\beta}}(\hat{t}_0) = & \eta_{\alpha\beta} + \frac{2M}{\bar{r}}\delta_{\alpha\beta} + \frac{M^2}{\bar{r}^2}(-2t_\alpha t_\beta + 3n_\alpha n_\beta) + \frac{1}{\bar{r}^2}t_{(\alpha}S_{\beta)j}n^j + O\left(\frac{1}{\bar{r}^3}\right) \\ & + \lambda \left[h_{\alpha\beta}^{\text{tail}} + 2A_{(\alpha,\beta)} + \frac{1}{r}K_{\alpha\beta} + \frac{\bar{t}}{\bar{r}^2}t_{(\alpha}\dot{S}_{\beta)j}n^j + O\left(\frac{1}{\bar{r}^2}\right) + \bar{t} O\left(\frac{1}{\bar{r}^3}\right) \right] \\ & + \lambda^2 \left[B_{\alpha i \beta j} \bar{x}^i \bar{x}^j + h_{\alpha\beta,\gamma}^{\text{tail}} \bar{x}^\gamma + M\mathcal{R}_{\alpha\beta}(\bar{x}^i) + 2B_{\alpha i \beta j} A^i \bar{x}^j + 2A_{(\alpha,\beta)\gamma} \bar{x}^\gamma \right. \\ & \left. + H_{\alpha\beta} + \frac{\bar{t}}{\bar{r}}\dot{K}_{\alpha\beta} + \frac{\bar{t}^2}{\bar{r}^2}t_{(\alpha}\ddot{S}_{\beta)j}n^j + O\left(\frac{1}{\bar{r}}\right) + \bar{t} O\left(\frac{1}{\bar{r}^3}\right) + \bar{t}^2 O\left(\frac{1}{\bar{r}^3}\right) \right] + O(\lambda^3) . \end{aligned} \quad (15)$$

The terms that are first order in λ in this equation satisfy the linearized vacuum Einstein equation about the background “near zone” metric (i.e., the terms that are 0th order in λ). From this equation, we find that $dS_{ij}/dt = 0$, i.e., to lowest order, spin is parallelly propagated along γ .

The terms that are second order in λ in this equation satisfy the linearized Einstein equation about the background “near zone” metric with source given by the second order Einstein tensor of the first order terms. Extracting the $\ell = 1$, electric parity, even-under-time-reversal part of this equation at $O(1/\bar{r}^2)$ and $O(\bar{t}/\bar{r}^3)$, we obtain (after considerable algebra!)

$$Z_{i,00} = \frac{1}{2M}S^{kl}R_{kl0i} - R_{0j0i}Z^j - \left(h_{i0,0}^{\text{tail}} - \frac{1}{2}h_{00,i}^{\text{tail}} \right) . \quad (16)$$

In other words, in the Lorenz gauge, the deviation vector field, Z^a , on γ that describes the first order perturbation to the motion satisfies

$$u^c \nabla_c (u^b \nabla_b Z^a) = \frac{1}{2M} R_{bcd}{}^a S^{bc} u^d - R_{bcd}{}^a u^b u^d Z^c - (g^{ab} + u^a u^b) (\nabla_a h_{bc}^{\text{tail}} - \frac{1}{2} \nabla_b h_{cd}^{\text{tail}}) u^c u^d . \quad (17)$$

Equation (17) gives the desired leading order corrections to motion along the geodesic γ . The first term on the right side of this equation is the Papapetrou “spin force”, which is the leading order “finite size” correction. The second term is just the usual right hand side of the geodesic deviation equation; it is not a correction to geodesic motion but rather allows for the possibility that the perturbation may make the body move along a different geodesic. Finally, the last term describes the gravitational self-force that we had sought to obtain, i.e., the corrections to the motion caused by the body’s self-field. Equation (17) gives the correct description of motion when the metric perturbation is in the Lorenz gauge. When the metric perturbation is expressed in a different gauge, the force will be different [4].

Although we have now obtained the perturbative correction to geodesic motion due to spin and self-force effects, at late times the small corrections due to self-force effects should accumulate (e.g., during an inspiral), and eventually the orbit should deviate significantly from the original, unperturbed geodesic γ . When this happens, it is clear our perturbative description in terms of a deviation vector defined on γ will not be accurate. Clearly, going to any (finite) higher order in perturbation theory will not help (much). However, if the mass and size of the body are sufficiently small, we expect that its motion is well described *locally* as a small perturbation of *some* geodesic. *Therefore, one should obtain a good description of the motion by making up (!) a “self-consistent perturbative equation”* that satisfies the following criteria: (1) It has a well posed initial value formulation. (2) It has the same “number of degrees of freedom” as the original system. (3) Its solutions correspond closely to the solutions of the original perturbation equation over a time interval where the perturbation remains small. In some sense, such a self-consistent perturbative equation would take into account the important (“secular”) higher order perturbative effects (to all orders), but ignore other higher order corrections. Such equations are commonly considered in physics. The MiSaTaQuWa equations appear to be a good candidate for a self-consistent perturbative equation associated with our perturbative result.

In summary, we have analyzed the motion of a small body or black hole in general relativity, assuming only the existence of a one-parameter family of solutions satisfying assumptions (i), (ii), and (iii) above. We showed that at lowest (“zeroth”) order, the motion of a “small” body is described by a geodesic, γ , of the “background” spacetime. We then derived a formula for the first order deviation of the “center of mass” worldline of the body from γ . The MiSaTaQuWa equations then arise as (candidate) “self-consistent perturbative equations” based on our first order perturbative result. Note that it is only at this stage that “phoney” linearized Einstein equations come into play.

We have recently applied this basic approach to the derivation of self-force in electromagnetism [5], and have argued that the reduced order form of the Abraham-Lorentz-Dirac equation provides an appropriate self-consistent perturbative equation associated with our first order perturbative result (whereas the original Abraham-Lorentz-Dirac equation is excluded). It should be possible to use this formalism to take higher order corrections to the motion into account in a systematic way in both the gravitational and electromagnetic cases.

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